

Symmetries and exact solutions of the rotating shallow water equations

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Abstract. Lie symmetry analysis is applied to study the nonlinear rotating shallow water equations. The 9-dimensional Lie algebra of point symmetries admitted by the model is found. It is shown that the rotating shallow water equations are related with the classical shallow water model with the change of variables. The derived symmetries are used to generate new exact solutions of the rotating shallow equations. In particular, a new class of time-periodic solutions with quasi-closed particle trajectories is constructed and studied. The symmetry reduction method is also used to obtain some invariant solutions of the model. Examples of these solutions are presented with a brief physical interpretation.

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1. Introduction

The article focuses on the Lie symmetry analysis of the rotating shallow water (RSW) model. In Cartesian frame of reference the RSW equations have the following form

$$\begin{aligned} u_t + uu_x + vu_y - fv + gh_x &= 0, \\ v_t + uv_x + vv_y + fu + gh_y &= 0, \\ h_t + (uh)_x + (vh)_y &= 0. \end{aligned} \tag{1.1}$$

Here (u, v) is the fluid velocity, h is the free surface height over the flat bottom, f is the constant Coriolis parameter and g is the constant gravity acceleration. The RSW model (1.1) arises from the three-dimensional rotating incompressible Euler equations (see [1], pp. 59–63) under the following assumptions: a) the scale H_0 of the vertical motion is much less than the scale L_0 of the horizontal motion, so that $H_0/L_0 \ll 1$; b) the fluid density is constant ($\rho = \text{const}$); c) the external force is due to gravity, and pressure obeys the hydrostatic approximation $p = pg(h - z) + p_0$ (p_0 is the constant pressure on the free surface); d) the axis of rotation of the fluid coincides with the vertical z -axis.

This nonlinear system of partial differential equations (PDEs) is widely-used approximation for atmospheric and oceanic motions in the midlatitudes with relatively large length and time scales. The model applies to phenomena that do not depend

substantially on temporal changes of the density stratification. In spite of its simplicity, it contains all essential ingredients of atmosphere and ocean dynamics at the synoptic scale. The main known mathematical results and physical applications concerning of the RSW model (1.1) as well as its linear theory, quasi-geostrophic limit and etc are presented in [1]–[4].

We recall here that the potential vorticity, defined by $\Omega = (v_x - u_y + f)h^{-1}$, is conserved on particles: $\Omega_t + u\Omega_x + v\Omega_y = 0$. Moreover, there are two important for applications non-dimensional parameters: Rossby number $\text{Ro} = U_0/(fL_0)$ and Froude number $\text{Fr} = U_0/\sqrt{gH_0}$ (here U_0 is the typical velocity scale for the flow). In non-dimensional form of the model Ro^{-1} and Fr^{-2} stand for f and g , correspondingly. In this study these parameters do not play a vital part. In fact, as result of obvious scaling transformations the system (1.1) can be reduced to the equivalent one with $f = 1$ and $g = 1$ (in case $f \neq 0$ and $g \neq 0$).

One of the methods for studying of PDEs is group analysis. This analytical approach based on symmetries of differential equations was originally introduced by Sophus Lie at the end-nineteenth century and further developed by Ovsyinnikov [5], Olver [6] and others. For each system of PDEs there is a symmetry group, that acts on the space of its independent and dependent variables, leaving the form of the system unchanged. The classical Lie symmetry analysis allows one to construct and study special types of analytical solutions of nonlinear PDEs in terms of solutions of lower-dimension equations. For modern description of the theory see, for example, [5]–[7]. Many applications of group analysis to PDEs are collected in [8]. The classical Lie method is an algorithmic procedure for which many symbolic manipulation programs were designed [9, 10]. This software became imperative in finding symmetries associated with large systems of PDEs.

Exact solutions of nonlinear systems descriptive of fluid motions with moving boundaries are uncommon even in the shallow water approximation. Freeman [11] and Sachdev [12] have obtained solutions for gravity wave motions under the assumption that the flow is a simple wave. It is interesting to note that the similarity solution obtained in [11] may be systematically generated by group methods [13]. In the latter work, symmetry analysis was often used to obtain analytical solutions to the nonlinear water-wave problems. Symmetries and exact solutions of a nonlinear system that models the finite motion of a rotating shallow liquid contained in a rigid basin were studied in [14] (see also [15]). At present there known many examples of exact (invariant and partial invariant) solutions for nonlinear models of fluid dynamics [13]–[27] and others. Amount of analytical solutions to the Euler equations govern the compressible inviscid flow were obtained within the framework of Ovsyinnikov’s “submodels programme” [16, 17].

In this paper we investigate Lie point symmetries and classes of exact solutions of the RSW equations. In Section 2, we determine the 9-dimension Lie algebra of infinitesimal transformations admitted by the RSW model. We also prove that the derived Lie algebra of symmetries is isomorphic to the Lie algebra of infinitesimal transformations admitted by (2+1)-dimensional shallow water (SW) equations. This

allows one to use its known optimal system of subalgebras. Moreover, we find the change of variables which transforms the RSW model to the SW equations. This is one of the principal results of the paper. In Section 3, we derive and analyse finite transformations corresponding to the nontrivial symmetries of the RSW model. On basis of these transformations, in Section 4, we construct new time-periodic exact solutions of the RSW equations. These solutions may be interpreted as pulsation of liquid volume under the influence of gravity and Coriolis forces. In Section 5, using some two-dimensional parameterized classes of the optimal system of subalgebras we reduce the RSW model to the ordinary differential equations (ODEs) and integrate them. In particular, we construct and study exact solutions describing rotational-symmetric subcritical and supercritical regimes of the stationary flow as well as various non-stationary regimes of spreading and collapse of a liquid ring.

2. Symmetries of the RSW equations and reduction to the SW model

2.1. Lie point symmetries of the RSW equations

On the basis of the group analysis of the differential equations we shall find the symmetry group of infinitesimal transformations associated with the RSW equations (1.1). Following [5], let us define infinitesimal generator X and its first prolongation Y

$$X = \xi^i(\mathbf{x}, \mathbf{u}) \partial_{x^i} + \eta^j(\mathbf{x}, \mathbf{u}) \partial_{u^j}, \quad Y = X + \zeta_i^j \partial_{u_i^j} \quad (i, j = 1, \dots, 3).$$

Here we use the following notations for convenience

$$\mathbf{x} = (x^1, x^2, x^3) = (t, x, y), \quad \mathbf{u} = (u^1, u^2, u^3) = (u, v, h)$$

$$\zeta_i^j = D_i \eta^j - u_i^j D_i \xi^j, \quad u_i^j = \frac{\partial u^j}{\partial x^i}, \quad D_i = \frac{\partial}{\partial x^i} + u_i^j \frac{\partial}{\partial u^j}.$$

To determine the group of symmetries admitted by the system of equations (1.1), we act on it with the first prolongation of generator X and go to the set of the system solutions. Consequently, we get a system of the determining equations for the unknown functions $\xi^i(\mathbf{x}, \mathbf{u})$ and $\eta^j(\mathbf{x}, \mathbf{u})$, that allows splitting in variables u_i^j . Let us omit the bulky intermediate calculations and present the final result of the symmetry group determination.

Theorem 1 *The symmetry group associated with the RSW equations (1.1) is generated by the following vector fields:*

$$\begin{aligned} X_1 &= \partial_x, \quad X_2 = \partial_y, \quad X_7 = \partial_t, \\ X_3 &= \cos(ft) \partial_x - \sin(ft) \partial_y - f \sin(ft) \partial_u - f \cos(ft) \partial_v, \\ X_4 &= \sin(ft) \partial_x + \cos(ft) \partial_y + f \cos(ft) \partial_u - f \sin(ft) \partial_v, \\ X_5 &= -y \partial_x + x \partial_y - v \partial_u + u \partial_v, \\ X_6 &= x \partial_x + y \partial_y + u \partial_u + v \partial_v + 2h \partial_h, \end{aligned} \tag{2.1}$$

$$\begin{aligned}
X_8 = & \cos(ft)\partial_t - \frac{f}{2}\left(x\sin(ft) - y\cos(ft)\right)\partial_x - \frac{f}{2}\left(x\cos(ft) + y\sin(ft)\right)\partial_y + \\
& + \frac{f}{2}\left((u - fy)\sin(ft) + (v - fx)\cos(ft)\right)\partial_u - \\
& - \frac{f}{2}\left((u + fy)\cos(ft) - (v + fx)\sin(ft)\right)\partial_v + fh\sin(ft)\partial_h, \tag{2.2}
\end{aligned}$$

$$\begin{aligned}
X_9 = & \sin(ft)\partial_t + \frac{f}{2}\left(x\cos(ft) + y\sin(ft)\right)\partial_x - \frac{f}{2}\left(x\sin(ft) - y\cos(ft)\right)\partial_y - \\
& - \frac{f}{2}\left((u - fy)\cos(ft) - (v - fx)\sin(ft)\right)\partial_u - \\
& - \frac{f}{2}\left((u + fy)\sin(ft) + (v + fx)\cos(ft)\right)\partial_v - fh\cos(ft)\partial_h. \tag{2.3}
\end{aligned}$$

Thus, the RSW equations (1.1) are invariant under translations in t , x and y (X_7 , X_1 and X_2 , correspondingly), helical rotations with respect to t (X_4 and X_5), rotation (X_5), scaling symmetry (X_6) and two complicated vector fields (X_8 and X_9) which will be discussed below. The infinitesimal symmetries (2.1)–(2.3) form Lie algebra L_9^f .

Symmetry properties of the RSW equations in Lagrangian coordinates were studied in [28]. Transition to Lagrangian variables is a nonlocal change of coordinate. Therefore Lie groups admitted by the model in Eulerian and Lagrangian variables do not completely coincide. In particular, there are no nontrivial symmetries defined by generators X_8 and X_9 in Lagrangian variables.

As mentioned above, in [14, 15] group analysis was applied to the model of rotating shallow liquid in a rigid basin with the following geometry

$$z = Z(x, y) = Ax^2 + By^2, \quad A > 0, \quad B > 0.$$

It was shown that in the elliptic paraboloid case, $A \neq B$, the symmetry algebra is spanned by six infinitesimal generators. In the circular paraboloid case, corresponding to $A = B$, the symmetry algebra is larger — namely, 9-dimensional. Note that these symmetries reduce to (2.1)–(2.3) for $A = B = 0$.

2.2. Properties of Lie algebra L_9^f

Let E be a given system of differential equations admitting a symmetry group G . The basic property of solutions for E is that any solution of the system E carried over by any transformation of the group G to a certain solution of the same system E . Therefore, two solutions of the system E are said to be essentially different with respect to G if they can not be transformed to each other by any transformation of the group G . Thus, it is useful to enumerate subgroups of the group G , which lead to essentially different solutions. By virtue of the fact that there are certain correspondences between the symmetry groups and Lie algebras, it is enough to construct an optimal system of subalgebras ΘL (see [5, 29, 30]), i.e. the minimal set of subalgebras of L , which exhaust all the essentially different invariant and partially invariant solutions to equations E .

In the present case there is no need to construct the optimal system of the subalgebras for Lie algebra L_9^f since we can use the results derived from the symmetry

analysis of the gas dynamic equations. In the case of zero Coriolis parameter ($f = 0$) the model under consideration reduces to the SW equation (or two-dimensional polytropic gas dynamic equations with polytropic exponent $\gamma = 2$)

$$\begin{aligned} u_t + uu_x + vv_y + gh_x &= 0, \\ v_t + uv_x + vv_y + gh_y &= 0, \\ h_t + (uh)_x + (vh)_y &= 0. \end{aligned} \tag{2.4}$$

The Lie algebra L_9 of infinitesimal symmetries of the SW equations is spanned by the following vector fields [5, 8]

$$\begin{aligned} Z_1 &= \partial_x, \quad Z_2 = \partial_y, \quad Z_3 = t\partial_x + \partial_u, \quad Z_4 = t\partial_y + \partial_v, \\ Z_5 &= -y\partial_x + x\partial_y - v\partial_u + u\partial_v, \\ Z_6 &= x\partial_x + y\partial_y + u\partial_u + v\partial_v + 2h\partial_h, \quad Z_7 = \partial_t, \\ Z_8 &= t^2\partial_t + tx\partial_x + ty\partial_y + (x - tu)\partial_u + (y - tv)\partial_v - 2th\partial_h, \\ Z_9 &= 2t\partial_t + x\partial_x + y\partial_y - u\partial_u - v\partial_v - 2h\partial_h. \end{aligned}$$

For this Lie algebra L_9 , the optimal system of subalgebras ΘL_9 , which includes 179 parameterized classes is presented in [18]. By virtue of the following theorem, parameterized classes of this optimal system of subalgebras can be applied for obtaining invariant or partially invariant solutions of the RSW equations (1.1).

Theorem 2 *Lie algebras $L_9(Z_1, \dots, Z_9)$ and $L_9^f(X_1, \dots, X_9)$ are isomorphic.*

Proof. Let us redefine the basis of generators of the Lie algebra L_9^f so that

$$\begin{aligned} Y_1 &= X_2 - X_4, \quad Y_2 = X_3 - X_1, \quad Y_3 = X_1 + X_3, \quad Y_4 = X_2 + X_4, \\ Y_5 &= X_5, \quad Y_6 = X_6, \quad Y_7 = \frac{1}{f} \left(X_7 - \frac{f}{2} X_5 - X_8 \right), \\ Y_8 &= \frac{1}{f} \left(X_7 - \frac{f}{2} X_5 + X_8 \right), \quad Y_9 = -\frac{2}{f} X_9. \end{aligned} \tag{2.5}$$

The commutation relations for the Lie algebra L_9^f in this basis are displayed in table 1. The basis is a canonical one chosen so that the Levi decomposition is immediately apparent from the commutation table. Thus $L_9^f = R \oplus N$, where $R = \{Y_1, \dots, Y_6\}$ and $N = \{Y_7, Y_8, Y_9\}$ constitute the maximal solvable ideal (radical) and the simple Lie algebra $sl(2)$. The nilradical of L_9^f is abelian and is generated by $\{Y_1, Y_2, Y_3, Y_4\}$.

It is easy to verify that the commutation relations for the Lie algebra $L_9(Z_1, \dots, Z_9)$ are defined by the same table of commutators (where symbols Y_k should be replaced by Z_k). The table of commutators for Lie algebras $L_9(Z_1, \dots, Z_9)$ and $L_9^f(Y_1, \dots, Y_9)$ are coincide completely. Consequently, Lie algebras L_9 and L_9^f are isomorphic. \square

Thus, to construct invariant or partially invariant solutions to the RSW equations (1.1) we can use generators Y_k defined by formulas (2.1)–(2.3) and (2.5) as well as the parameterized classes of the optimal system of the subalgebras introduced in [18].

Table 1. Table of commutators for Lie algebra $L_9^f(Y_1, \dots, Y_9)$.

	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_7	Y_8	Y_9
Y_1	0	0	0	0	Y_2	Y_1	0	Y_3	Y_1
Y_2	0	0	0	0	$-Y_1$	Y_2	0	Y_4	Y_2
Y_3	0	0	0	0	Y_4	Y_3	$-Y_1$	0	$-Y_3$
Y_4	0	0	0	0	$-Y_3$	Y_4	$-Y_2$	0	$-Y_4$
Y_5	$-Y_2$	Y_1	$-Y_4$	Y_3	0	0	0	0	0
Y_6	$-Y_1$	$-Y_2$	$-Y_3$	$-Y_4$	0	0	0	0	0
Y_7	0	0	Y_1	Y_2	0	0	0	Y_9	$2Y_7$
Y_8	$-Y_3$	$-Y_4$	0	0	0	0	$-Y_9$	0	$-2Y_8$
Y_9	$-Y_1$	$-Y_2$	Y_3	Y_4	0	0	$-2Y_7$	$2Y_8$	0

2.3. Reduction to the SW model

In view of the previous theorem arises important question. Is there change of variables such that each generator Y_k ($k = 1, \dots, 9$) reduces to generator Z_k ? In the case of existence one it is necessary to verify if the RSW equations (1.1) reducible to the SW equations (2.4) under this transformation. The answer on the question is given by the following theorem, which can be easy proved by straightforward calculations.

Theorem 3 *The models (1.1) and (2.4) as well as their symmetries are related by the following transformation*

$$\begin{aligned}
 t' &= -\frac{1}{f} \cot\left(\frac{ft}{2}\right), & x' &= -\frac{1}{2}\left(x \cot\left(\frac{ft}{2}\right) - y\right), & y' &= -\frac{1}{2}\left(x + y \cot\left(\frac{ft}{2}\right)\right), \\
 u' &= -\frac{1}{2}\left(u \sin(ft) - v(1 - \cos(ft)) - fx\right), \\
 v' &= -\frac{1}{2}\left(u(1 - \cos(ft)) + v \sin(ft) - fy\right), & h' &= \frac{h}{2}\left(1 - \cos(ft)\right).
 \end{aligned} \tag{2.6}$$

In fact, let us suppose that a set of functions $u(t, x, y)$, $v(t, x, y)$, $h(t, x, y)$ satisfies to the RSW equations (1.1). Then the immediate calculations show that the following set of functions $u'(t', x', y')$, $v'(t', x', y')$, $h'(t', x', y')$, given by formulae (2.6), satisfies to the SW equations (2.4). Thus, the models (1.1) and (2.4) are related by the change of variables (2.6).

It is easy to clear up that each generator Y_k in the variables with prime (2.6) takes form Z_k . More precisely, $Y'_k = Z_k$ ($k = 1, 2, 5, 6, 9$), $Y'_k = fZ_k$ ($k = 3, 4, 8$), $Y'_7 = f^{-1}Z_7$. Using scaling it is possible to fix formulae (2.6) (or reduce (1.1) to the equivalent one with $f = 1$) in order to $Y'_k = Z_k$ for all $k = 1, \dots, 9$ without coefficient f . However, this form is more convenient as it conserve physical dimension of the variables.

Note that the mapping (2.6) transforms any solution of the SW equations (2.4) defined for all time moments ($-\infty < t < \infty$) to the solution of the RSW equations (1.1) defined only for bounded time interval ($0 < t < 2\pi/f$) or, may be, to the stationary

solution. This change of variables allows one to construct solutions of the RSW model using known solutions of the SW equations and vice versa.

For example, let us consider constant solution of the SW equations (2.4): $u = u_0$, $v = v_0$, $h = h_0$. Obviously, in this case trajectories of fluid particles are lines. According to (2.6) we obtain the following solution of the RSW model (1.1)

$$\begin{aligned} u &= u_0 \cot\left(\frac{ft}{2}\right) - v_0 + \frac{f}{2}\left(x \cot\left(\frac{ft}{2}\right) + y\right), \\ v &= u_0 - v_0 \cot\left(\frac{ft}{2}\right) - \frac{f}{2}\left(x - y \cot\left(\frac{ft}{2}\right)\right), \quad h = \frac{2h_0}{1 - \cos(ft)}. \end{aligned} \quad (2.7)$$

Solution (2.7) is defined over the finite time interval $0 < t < 2\pi/f$. Note that particle trajectories have the following form (after time elimination)

$$(x - A)^2 + (y - B)^2 = R^2,$$

where

$$A = x_0/2 + u_0/f, \quad B = y_0/2 + v_0/f, \quad R^2 = (-x_0/2 + u_0/f)^2 + (y_0/2 - v_0/f)^2.$$

Here x_0 and y_0 are coordinates of the particle at $t = \pi/f$. Thus, each fluid particle moves around a circumference. However, these trajectories do not close because solution becomes invalid as time t tends to 0 or $2\pi/f$. This solution describes essentially two-dimension fluid flow. In the case $u_0^2 + v_0^2 = 0$ it reduces to the rotationally symmetric solution in the polar frame of reference (3.2)

$$U = \frac{fr}{2} \cot\left(\frac{ft}{2}\right), \quad V = -\frac{fr}{2}, \quad h = \frac{2h_0}{1 - \cos(ft)}.$$

Inversely, let us take solution $u = 0$, $v = 0$, $h = h_0$ of the RSW equations (1.1), corresponding to the state of rest. Transformation (2.6) yields the following rotationally-symmetric solution of the SW equations (2.4)

$$U = \frac{f^2 tr}{1 + f^2 t^2}, \quad V = \frac{fr}{1 + f^2 t^2}, \quad h = \frac{h_0}{1 + f^2 t^2}$$

(primes are omitted here).

Using analogy with gas dynamic, we can conclude that these solutions belong to the class of “barochronous” [19, 20] motions (as depth h depends only on time t).

3. Finite transformations corresponding to the “complicate” generators

It is well known that generator $X = \xi^i \partial_{x^i}$ can be associated with one-parameter group of transformation G_1 given by the finite relations of the form $\bar{x}^i = \bar{x}^i(x^1, \dots, x^N, a)$, where a is a real-valued parameter. To determine these relations we shall solve the system of Lie equations

$$\frac{\partial \bar{x}^i}{\partial a} = \xi^i(\bar{x}^1, \dots, \bar{x}^N), \quad \bar{x}^i|_{a=0} = x^i \quad (i = 1, \dots, N). \quad (3.1)$$

Let us find the finite transformations corresponding to the nontrivial infinitesimal generators Y_7 , Y_8 and Y_9 . For the sake of convenience we shall use polar coordinates

(r, θ) , radial U and circular V velocity vector components related to (x, y) coordinates and (u, v) velocities, so that

$$x = r \cos \theta, \quad y = r \sin \theta, \quad u = U \cos \theta - V \sin \theta, \quad v = U \sin \theta + V \cos \theta. \quad (3.2)$$

In these variables admissible symmetries Y_7, Y_8, Y_9 take the following form

$$\begin{aligned} \hat{Y}_7 = & f^{-1}(1 - \cos(ft))\partial_t + 2^{-1}r \sin(ft)\partial_r - 2^{-1}(1 - \cos(ft))\partial_\theta - \\ & - 2^{-1}(U \sin(ft) - fr \cos(ft))\partial_U - 2^{-1}(V + fr) \sin(ft)\partial_V - h \sin(ft)\partial_h, \end{aligned}$$

$$\hat{Y}_8 = -\hat{Y}_7 + 2f^{-1}\partial_t - \partial_\theta,$$

$$\begin{aligned} \hat{Y}_9 = & -2f^{-1} \sin(ft)\partial_t - r \cos(ft)\partial_r + \sin(ft)\partial_\theta + \\ & + (U \cos(ft) + fr \sin(ft))\partial_U + (V + fr) \cos(ft)\partial_V + 2h \cos(ft)\partial_h. \end{aligned}$$

Integration of the Lie equations (3.1) associated with generator \hat{Y}_9 gives the following result ($t \neq (2n+1)\pi/f$, n — integer):

$$\begin{aligned} \bar{t} &= \frac{2}{f} \arctan(\alpha\tau) + \chi(t), \quad \bar{r} = r \sqrt{\frac{\alpha(1+\tau^2)}{1+\alpha^2\tau^2}}, \\ \bar{\theta} &= \theta + \arctan(\tau) - \arctan(\alpha\tau), \\ \bar{U} &= \left(U - \frac{fr(\alpha^2-1)\tau}{2(1+\alpha^2\tau^2)} \right) \sqrt{\frac{1+\alpha^2\tau^2}{\alpha(1+\tau^2)}}, \\ \bar{V} &= \left(V + \frac{fr(\alpha-1)(\alpha\tau^2-1)}{2(1+\alpha^2\tau^2)} \right) \sqrt{\frac{1+\alpha^2\tau^2}{\alpha(1+\tau^2)}}, \quad \bar{h} = h \frac{1+\alpha^2\tau^2}{\alpha(1+\tau^2)}. \end{aligned} \quad (3.3)$$

Here $\tau = \tan(\frac{ft}{2})$, $\alpha = \exp(-2a)$, $\chi(t) = 2\pi k/f$, where integer k is such that value t specified by initial condition at $a = 0$ belongs to interval $((2k-1)\pi/f, (2k+1)\pi/f)$. In the case $t = (2n+1)\pi/f$ solution of the equations (3.1) have the following form

$$\begin{aligned} \bar{t} &= t = (2n+1)\pi/f, \quad \bar{r} = r/\sqrt{\alpha}, \quad \bar{\theta} = \theta, \\ \bar{U} &= U\sqrt{\alpha}, \quad \bar{V} = (V + (\alpha-1)(2\alpha)^{-1}fr)\sqrt{\alpha}, \quad \bar{h} = \alpha h. \end{aligned} \quad (3.4)$$

Let us show that functions (3.3) can be defined at points $t = t_* = (2n+1)\pi/f$ by continuity in compliance with (3.4). The derived mapping will be continuously differentiable with respect to all its arguments. Let us investigate one-sided limits in formula (3.3) at $t \rightarrow t_* \mp 0$ (the remaining variables are fixed). In this case $\tau(t) = \tan(\frac{ft}{2}) \rightarrow \pm\infty$. Discontinuous at point $t = t_*$ function $\frac{2}{f} \arctan(\alpha\tau(t))$ has the following limits: π/f on the left-hand side and $-\pi/f$ on the right-hand side of the discontinuity. Function $\chi(t)$ is piecewise constant and possesses the value of $2\pi n/f$ on the left-hand side and $2\pi(n+1)/f$ on the right-hand side of point $t = t_*$. Thus, function $\bar{t}(t, \alpha) = \frac{2}{f} \arctan(\alpha\tau) + \chi(t)$ with $t \rightarrow t_* \mp 0$ has one-sided limits which coincide and are equal to $(2n+1)\pi/f$. This agrees with the first formula in (3.4). Correspondence of the remain functions in formulae (3.3) and (3.4) is easily established by calculation of their one-side limits at $t = t_*$. Moreover, straightforward analysis shows that the mapping (3.3) redefined at points $t = t_* = (2n+1)\pi/f$ in compliance with formulae (3.4) is continuously differentiable with respect to all the arguments.

Similarly we calculate finite transformations corresponding to the generator \hat{Y}_8 :

$$\begin{aligned}\bar{t} &= \frac{2}{f} \arctan(\tau + a) + \chi(t), \quad \bar{r} = r \sqrt{\frac{\tau^2 + 1}{(\tau + a)^2 + 1}}, \\ \bar{\theta} &= \theta + \arctan(\tau) - \arctan(\tau + a), \\ \bar{U} &= \left(U + \frac{fr}{2} \frac{(\tau^2 + a\tau - 1)a}{(\tau + a)^2 + 1} \right) \sqrt{\frac{(\tau + a)^2 + 1}{\tau^2 + 1}}, \\ \bar{V} &= \left(V + \frac{fr}{2} \frac{(2\tau + a)a}{(\tau + a)^2 + 1} \right) \sqrt{\frac{(\tau + a)^2 + 1}{\tau^2 + 1}}, \quad \bar{h} = h \frac{(\tau + a)^2 + 1}{\tau^2 + 1}.\end{aligned}\tag{3.5}$$

Here τ and χ are the same functions as before.

Finite transformations corresponding to the generator \hat{Y}_7 have the same form (3.5), where $\sigma = -\cot(\frac{ft}{2})$ and $\chi_1(t)$ stand for τ and $\chi(t)$. Function $\chi_1(t)$ is equal to $(2k + 1)\pi/f$, where integer k is such that value t specified by initial condition in the system (3.1) at $a = 0$ belongs to interval $(2k\pi/f, 2(k + 1)\pi/f)$.

According to the general theory [5, 6] governing equations (1.1) presented in the polar coordinates (3.2)

$$\begin{aligned}\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial r} + \frac{V}{r} \frac{\partial U}{\partial \theta} - \frac{V^2}{r} - fV + g \frac{\partial h}{\partial r} &= 0, \\ \frac{\partial V}{\partial t} + U \frac{\partial V}{\partial r} + \frac{V}{r} \frac{\partial V}{\partial \theta} + \frac{UV}{r} + fU + \frac{g}{r} \frac{\partial h}{\partial \theta} &= 0, \\ \frac{\partial h}{\partial t} + \frac{1}{r} \frac{\partial(rUh)}{\partial r} + \frac{1}{r} \frac{\partial(Vh)}{\partial \theta} &= 0\end{aligned}\tag{3.6}$$

remain unchanged under the substitutions (3.3) and (3.5) corresponding to the finite transformations of the “complicate” generators. This fact allows one to formulate the following theorem based on the finite transformations (3.3). Similar theorem can be obtained using formulae (3.5).

Theorem 4 *If a set of functions*

$$U = \bar{U}(t, r, \theta), \quad V = \bar{V}(t, r, \theta), \quad h = \bar{h}(t, r, \theta)$$

satisfies the system of equations (3.6) then the following set of functions

$$\begin{aligned}U(t, r, \theta) &= \left(\bar{U}(\bar{t}, \bar{r}, \bar{\theta}) + \frac{f\bar{r}}{2} \frac{(\alpha^2 - 1)\bar{\tau}}{\alpha^2 + \bar{\tau}^2} \right) \sqrt{\frac{\alpha^2 + \bar{\tau}^2}{\alpha(1 + \bar{\tau}^2)}}, \\ V(t, r, \theta) &= \left(\bar{V}(\bar{t}, \bar{r}, \bar{\theta}) - \frac{f\bar{r}}{2} \frac{(\alpha - 1)(\bar{\tau}^2 - \alpha)}{\alpha^2 + \bar{\tau}^2} \right) \sqrt{\frac{\alpha^2 + \bar{\tau}^2}{\alpha(1 + \bar{\tau}^2)}}, \\ h(t, r, \theta) &= \frac{\alpha^2 + \bar{\tau}^2}{\alpha(1 + \bar{\tau}^2)} \bar{h}(\bar{t}, \bar{r}, \bar{\theta})\end{aligned}\tag{3.7}$$

also satisfies the same system. Here

$$\begin{aligned}\tau &= \tan\left(\frac{ft}{2}\right), \quad \bar{\tau} = \tan\left(\frac{f\bar{t}}{2}\right), \quad \bar{t} = \frac{2}{f} \arctan(\alpha\tau) + \chi(t), \\ \bar{r} &= r \sqrt{\frac{\alpha(1 + \tau^2)}{1 + \alpha^2\tau^2}}, \quad \bar{\theta} = \theta + \arctan(\tau) - \arctan(\alpha\tau),\end{aligned}\tag{3.8}$$

where α is an arbitrary positive constant, $\chi(t)$ is the piecewise constant function which is equal to $2\pi k/f$ on the interval $t \in ((2k-1)\pi/f, (2k+1)\pi/f)$, k — integer. We assume that $\bar{t} = t$, $\bar{r} = r/\sqrt{\alpha}$, $\bar{\theta} = \theta$ at points $t = (2k+1)\pi/f$.

4. Time-periodic exact solutions of the RSW equations

The above-formulated theorem makes it possible to obtain new exact solutions of the RSW model by using known ones. Similar approach was applied to the gas dynamic equations in [21, 22]. Obviously, equations (3.6) have the following class of stationary rotationally-symmetric solutions

$$U = \bar{U} = 0, \quad V = \bar{V}(r), \quad h = \bar{h}(r) = \frac{1}{g} \int_0^r \left(\frac{\bar{V}^2}{r} + f\bar{V} \right) dr + h_0, \quad (4.1)$$

where $\bar{V}(r)$ is an arbitrary smooth function, h_0 is a positive constant. According to the preceding theorem and formulae (3.7) the functions

$$\begin{aligned} U &= \frac{fr}{2} \frac{(\alpha^2 - 1)\tau}{1 + \alpha^2\tau^2}, \\ V &= \sqrt{\frac{\alpha(1 + \tau^2)}{1 + \alpha^2\tau^2}} \bar{V} \left(r \sqrt{\frac{\alpha(1 + \tau^2)}{1 + \alpha^2\tau^2}} \right) - \frac{fr}{2} \frac{(\alpha - 1)(\alpha\tau^2 - 1)}{1 + \alpha^2\tau^2}, \\ h &= \frac{\alpha(1 + \tau^2)}{1 + \alpha^2\tau^2} \bar{h} \left(r \sqrt{\frac{\alpha(1 + \tau^2)}{1 + \alpha^2\tau^2}} \right); \quad \tau = \tan \left(\frac{ft}{2} \right) \end{aligned} \quad (4.2)$$

is the solution of the equations (3.6). The principal prerequisite to the existence of periodic solutions of the form (4.2) is the presence of nonzero Coriolis parameter f . Let us consider in detail two typical solutions from class (4.2).

4.1. Example #1: Pulsation of the liquid cylinder

Let us consider solution from class (4.1) corresponding to the state of rest: $\bar{U} = 0$, $\bar{V} = 0$, and $\bar{h} = h_0$. By substitution of these functions into (4.2), the time-periodic exact solution of the RSW equations (3.6)

$$\begin{aligned} U &= \frac{fr}{2} \frac{(\alpha^2 - 1)\tau}{1 + \alpha^2\tau^2}, \quad V = -\frac{fr}{2} \frac{(\alpha - 1)(\alpha\tau^2 - 1)}{1 + \alpha^2\tau^2}, \\ h &= \frac{\alpha(1 + \tau^2)h_0}{1 + \alpha^2\tau^2} \end{aligned} \quad (4.3)$$

is obtained. Here, as before, $\tau = \tan(\frac{ft}{2})$. Particle trajectories are defined by the system of equations

$$\frac{dr}{dt} = U, \quad \frac{d\theta}{dt} = \frac{V}{r}, \quad r|_{t=0} = r_0, \quad \theta|_{t=0} = \theta_0 \quad (4.4)$$

and in this case have the following form

$$r(t) = r_0 \sqrt{\frac{1 + \alpha^2\tau^2(t)}{1 + \tau^2(t)}}, \quad \theta(t) = \theta_0 + \arctan(\alpha\tau(t)) - \arctan(\tau(t)). \quad (4.5)$$

Formulae (4.5) can be rewritten in a simpler form

$$(x - A)^2 + (y - B)^2 = R^2,$$

where

$$A = 2^{-1}(\alpha + 1)r_0 \cos \theta_0, \quad B = 2^{-1}(\alpha + 1)r_0 \sin \theta_0, \quad R = 2^{-1}(\alpha - 1)r_0.$$

Thus, each fluid particle moves round a circumference of radius R , which is proportional to the initial distance of the particle from the origin of the system of coordinates. All particles return to their initial positions at time $T = 2\pi/f$, that is period of solution (4.3). The solution can be interpreted as pulsation of the liquid cylinder with impermeable boundary moving in accordance with the first formula (4.5), where r_0 is the initial cylinder boundary radius. The height of the liquid column, depending only on time, is given by last formula in (4.3) (αh_0 is fluid surface height at initial time $t = 0$). We also note that time-periodic solution (4.3) is the solution with constant potential vorticity $\Omega = f/h_0$.

4.2. Example #2: Pulsation of the liquid “drop”

Let us consider the following solution from class (4.1)

$$\bar{U} = 0, \quad \bar{V}(r) = lr^2, \quad \bar{h}(r) = \frac{l^2}{4g} \left(r^4 + \frac{4f}{3l} r^3 + \frac{f^4}{3l^4} \right), \quad (4.6)$$

where $l = -f^2 \sqrt{\alpha/(12g)} < 0$. In this case the fluid is localized on the rotating plane within the cylinder $0 \leq r \leq -f/l$ (we suppose, that $f > 0$). The depth of fluid h is maximal at $r = 0$ and vanishes at $r = -f/l$. According to formulae (4.2) and (4.6) we arrive to the new time-periodic exact solution of the RSW equations (3.6), which describes pulsation of liquid volume under the influence of gravitation and rotation.

Integrating the equations of trajectories (4.4) with the functions U , V defined by (4.2) and (4.6), we obtain the following result

$$r(t) = r_0 \sqrt{\frac{1 + \alpha^2 \tau^2(t)}{1 + \tau^2(t)}}, \quad (4.7)$$

$$\theta(t) = (2Cf^{-1} + 1) \arctan(\alpha\tau(t)) - \arctan(\tau(t)) + C\chi(t) + \theta_0.$$

Here $C = lr_0\sqrt{\alpha}$; $\chi(t)$ is a piecewise constant function which takes the value $2\pi k/f$ for t from the interval $((2k-1)\pi/f, (2k+1)\pi/f)$, where k is an integer. Constants r_0 and θ_0 specify the position of the particle on the plane at initial time $t = 0$.

Figure 1 presents the fluid depth depending on the radius r at different time moments $t = \pi n/(2f)$ with $n = 0, 1, 2$ (lines 0, 1, and 2, correspondingly). The fluid depth takes zero value at the circumference of radius

$$r = R_*(t) = -\frac{f}{l} \sqrt{\frac{1 + \alpha^2 \tau^2(t)}{\alpha(1 + \tau^2(t))}}.$$

The graphs shown in figure 1 and others are plotted for $\alpha = 2$, $f = 1$, $g = 1$. As the parameters change, the qualitative view of the figure remains the same.

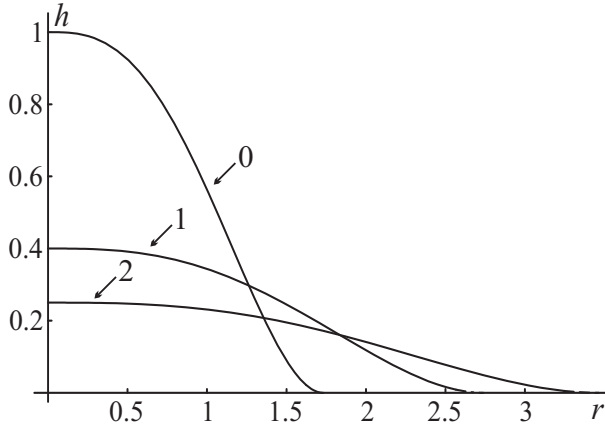


Figure 1. The fluid depth $h(t, r)$ at $t = \pi n/(2f)$, $n = 0, 1, 2$ (lines 0, 1, and 2, correspondingly).

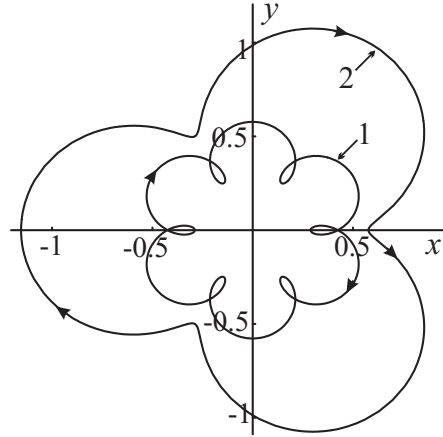


Figure 2. Typical trajectories of the fluid particles motion.

Formulae (4.7) imply that all particles, which belong to the circumference of radius $r_0 = -\frac{f}{l\sqrt{\alpha}} \frac{m}{M}$ (here $M \geq m > 0$ are integers) at initial time $t = 0$, have closed trajectories. At $t = 2\pi M/f$ the particles return to their starting position at $t = 0$. Generally, trajectories of fluid particles are quasi-closed. This means that for any specified value $\varepsilon > 0$ it is possible to choose $t_\varepsilon > 0$, so that the particle at $t = t_\varepsilon$ will be located at a distance of no more than ε from its initial position at $t = 0$. It should be pointed out that within the period of the solution $T = 2\pi/f$ only those particles, which belong to the boundary $r = R_*(t)$, where fluid depth h vanishes, return to their starting positions. In figure 2 closed trajectories of particles, located at $t = 0$ in the circumference of radius $r = 1/(2\sqrt{3})$ (curve 1) and $r = 1/\sqrt{3}$ (curve 2), are shown. The graphics correspond to the choice $(m, M) = (1, 6)$ and $(1, 3)$, with the above-indicated values α , f and g . Arrows show the direction of the particles motion.

Figure 3 shows the evolution of the material curves, which consist always of the same fluid particles. In contrast to the previous solution (4.3), here an arbitrary closed material curve, except for circumferences with centre at origin of coordinates, never returns to its initial position. Propagation of the closed material curve (which is initially the circumference of radius $r = 0.3$ with its centre at point $(x, y) = (0.4, 0.5)$) is shown in figure 3(a) at the instants $t = \pi n/(2f)$ with $n = 0, \dots, 4$ (curves 0–4). The trajectory of the fluid particle, initially located at point $(x, y) = (0.7, 0.5)$, is shown in figure 3(a) by dashed line. As time growth this material curve transforms into helix. Figure 3(b) presents two material curves which form circumferences at $t = 0$ (labeled as 1 and 2); and these material curves at the instant $t = 7\pi/f$ (helixes 3 and 4, correspondingly).

Remark 1 The deformation of the material curve, shown in the Fig. 3, is typical for all solutions from the class (4.2) where function \bar{V} is not identically zero. Indeed, we can integrate equations of trajectories (4.4) and represent the solution in the form (4.7) with $C = (r_0\sqrt{\alpha})^{-1}\bar{V}(r_0\sqrt{\alpha})$.

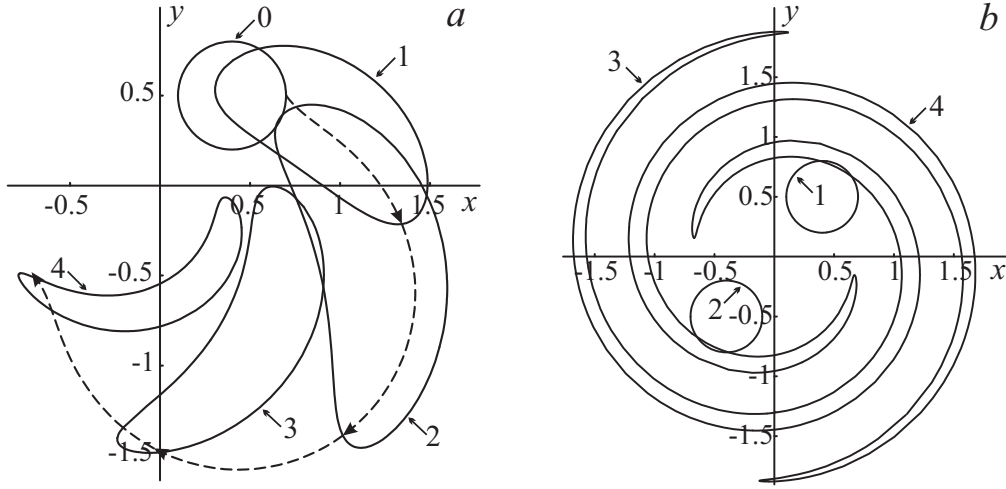


Figure 3. The evolution of the material curves: *a* — short time behaviour; *b* — long time behaviour.

5. Rotationally-symmetric flows

The infinitesimal symmetries (2.1)–(2.3) allow us to construct invariant and partially invariant solutions to the RSW model. This section is devoted to a set of rotationally-symmetric submodels, which can be reduced to ODEs. These submodels are derived by invariant reduction of the RSW equations (3.6) with the use of two-dimensional parameterized classes of the optimal system of the subalgebra ΘL_9 . Therefore, the obtained solutions are essentially different, that is one solution does not reduce to the other by change of the variables. Note that these solutions can be obtained from the corresponding invariant solutions of the SW equations (2.4) using transformation (2.6).

5.1. Stationary rotationally-symmetric flows in a ring

Submodel

$$\begin{aligned} UU_r - r^{-1}V^2 - fV + gh_r &= 0, \\ U(V_r + r^{-1}V + f) &= 0, \quad (rUh)_r = 0. \end{aligned} \quad (5.1)$$

characterizes the class of stationary rotationally-symmetric solutions. Equations (5.1) are derived according to the invariant reduction of the RSW model (3.6) with the use of subalgebra $(Y_5, Y_7 + Y_8)$ of admitted infinitesimal symmetries (in the polar coordinates the generators have the following form $\hat{Y}_5 = \partial_\theta$ and $\hat{Y}_7 + \hat{Y}_8 = 2f^{-1}\partial_t - \partial_\theta$).

Assuming $U = 0$, integration of (5.1) yields (4.1). In case $U \neq 0$, we obtain the following result

$$U = \frac{C_3}{rh}, \quad V = \frac{C_2}{r} - \frac{fr}{2}, \quad F(h, r) = h^3 + \varphi_1(r)h^2 + \varphi_2(r) = 0, \quad (5.2)$$

where C_i ($i = 1, 2, 3$) are constants and

$$\varphi_1(r) = \frac{1}{g} \left(\frac{f^2 r^2}{8} + \frac{C_2^2}{2r^2} - C_1 \right), \quad \varphi_2(r) = \frac{C_3^2}{2gr^2}.$$

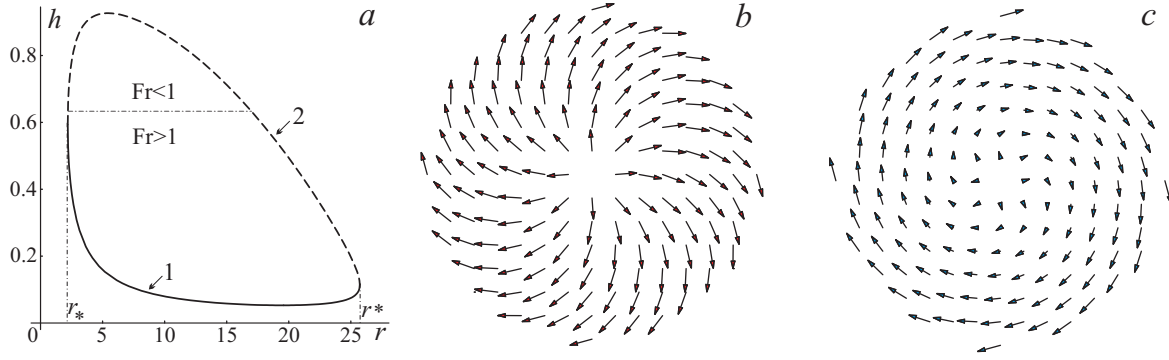


Figure 4. Stationary rotationally-symmetric flow in a ring bounded by characteristics: *a* — free surface height (line 1 corresponds to supercritical flow, line 2 — subcritical flow); *b* — velocity field (supercritical flow); *c* — velocity field (subcritical flow).

The necessary condition of existence of solutions to the equation $F(h, r) = 0$ with $h > 0$ is inequality $C_1 > |C_2|/2$, which guarantees that $\varphi_1 < 0$ for some values r .

The straightforward analysis shows that equation $F(h, r) = 0$ has two branches of solutions $h = h(r)$ defined within the interval $r_* \leq r \leq r^*$ (see figure 4(a)). At the endpoints $r = r_*$ and $r = r^*$ the derivative F_h vanishes: $F_h = (3h + 2\varphi_1)h = 0$. Hence, function $h(r)$ at the endpoints is finite $h = h_c$ (here $h_c = -\frac{2}{3}\varphi_1(r_*)$ or $h_c = -\frac{2}{3}\varphi_1(r^*)$), whereas its derivative is unbounded: $h'(r_*) = h'(r^*) = -F_r/F_h = \infty$. The relation $U^2/2 + gh = -g\varphi_1(r)$ is fulfilled for the considered solution (5.2), so that we get equality $U^2 = gh$ in circumferences of radii $r = r_*$ and $r = r^*$. Thus, the surfaces $r = r_*$ and $r = r^*$ are sonic characteristics of the model. Recall that the surface $\Phi(t, r, \theta) = \text{const}$ is the sonic characteristic of equations (3.6) on the solution (U, V, h) if function $\Phi(t, r, \theta)$ satisfies to equation $\Phi_t + U\Phi_r + r^{-1}V\Phi_\theta = \pm\sqrt{gh}$.

The flow is supercritical (subcritical) in case $\text{Fr} > 1$ ($\text{Fr} < 1$), correspondingly. Here $\text{Fr} = q/\sqrt{gh}$ is Froude number and $q = \sqrt{U^2 + V^2}$ is the magnitude of the velocity vector. It follows from (5.2) that $q^2 = 2C_1 - C_2f - 2gh$. Hence, $\text{Fr} = 1$ at $h = h_s = (2C_1 - C_2f)/(3g)$. One can easily prove that $h_c \leq h_s$. Therefore, the lower branch of solution to equation $F(h, r) = 0$ corresponds to the supercritical flow. In figure 4(a) are presented the lower (line 1) and the upper (line 2) branches of solution to equation $F(h, r) = 0$. In figures 4(b) and 4(c) the velocity fields (u, v) are shown for lower branch of free surface height $h = h(r)$ and upper branch, correspondingly. The graphics are derived for $C_i = 1$, $g = 1$, $f = 0.1$. As the parameters change, the qualitative view of the graphics remains the same.

It is interesting to note that the class of stationary rotationally-symmetric solutions of the RSW equations corresponds to the class of solutions of the SW model which is invariant under rotation (generator Z_5) and nontrivial projective transformation (generator $Z_7 + Z_8$). On the basis of projective transformation exact invariant solutions of the SW model and two-dimension gas dynamic equations (with special state equation) were obtain and studied in [21, 24, 23] and others.

5.2. Collapse of a liquid ring. Regime #1

Let us consider the submodel derived by the invariant reduction of the RSW equations (3.6) using two-dimensional subalgebra (Y_5, Y_7) of infinitesimal transformations. According to the algorithm of invariant solutions construction, let us solve the equations

$$\hat{Y}_5 J = 0, \quad \hat{Y}_7 J = 0$$

to obtain the set of basic invariants

$$J = \left\{ \frac{1 - \cos(ft)}{r^2}, \quad rU - \frac{fr^2}{2} \frac{\sin(ft)}{1 - \cos(ft)}, \quad rV + \frac{fr^2}{2}, \quad r^2 h \right\}.$$

The representation of an invariant solution is

$$U = \frac{\varphi(\lambda)}{r} + \frac{fr}{2} \cot\left(\frac{ft}{2}\right), \quad V = \frac{\psi(\lambda)}{r} - \frac{fr}{2}, \quad h = \frac{\eta(\lambda)}{r^2}, \quad (5.3)$$

where $\lambda = (1 - \cos(ft))r^{-2}$. By substituting the representation of solution into (3.6), one obtains the system of ODEs for unknowns φ, ψ, η

$$(\lambda(\varphi^2 + 2g\eta))' + \psi^2 = 0, \quad \lambda\varphi\psi' = 0, \quad (\lambda\varphi\eta)' = 0. \quad (5.4)$$

By virtue of the second equation (5.4) either $\varphi = 0$ or $\psi = \text{const}$. In the first case we get the following class of solutions

$$\varphi = 0, \quad \psi = \psi(\lambda), \quad \eta = \frac{1}{\lambda} \left(\lambda_0 \eta_0 - \frac{1}{2g} \int_{\lambda_0}^{\lambda} \psi^2(\nu) d\nu \right). \quad (5.5)$$

Here $\psi(\lambda)$ is an arbitrary smooth function, and η_0, λ_0 are constants. Solution (5.3), (5.5) is defined over the finite time interval $t \in [t_0, 2\pi/f)$, $t_0 > 0$. It has a singularity at $r \rightarrow \infty$. In some cases, depending on the choice of function ψ , the solution has singularity at $r \rightarrow 0$.

Notice, that surfaces $\lambda(t, r) = \text{const}$ are contact characteristics on the solution (5.3), (5.5). This allows one to interpret the solution as a collapse of a liquid ring (or a liquid cylinder) $R_1(t) \leq r \leq R_2(t)$ compressed by pistons which move according to the law

$$R_i(t) = R_{i0} \sin(ft/2) \quad (0 \leq R_{10} < R_{20}),$$

so that the pistons surfaces coincide with the contact characteristics. In the course of the collapse of a liquid ring (with $t \rightarrow 2\pi/f$), the radial velocity $U = \frac{fr}{2} \cot\left(\frac{ft}{2}\right)$ in the range of $R_1(t) \leq r \leq R_2(t)$ is bounded, whereas the liquid depth h as well as the circular velocity V (with any $\psi \neq \text{const}$) increases infinitely.

The other class of solutions of submodel (5.1) is derived in case $\varphi \neq 0$, $\psi = C_2 = \text{const}$. Whereas, $\eta(\lambda) = C_3/(\lambda\varphi)$ and function $\varphi(\lambda)$ are defined from cubic equation $\varphi^3 + (C_2^2 - C_1/\lambda)\varphi + 2gC_2/\lambda = 0$. The solution of the RSW equations (3.6) is obtained by using representation of solution (5.3).

These unsteady solutions of the RSW equations can be also derived from well-known stationary rotational-symmetric solutions of the SW equations using transformations (2.6) which relate the models.

5.3. Collapse of a liquid ring. Regime #2

Submodel

$$\varphi' = (\psi + f)\psi - \varphi^2 - 2g\eta, \quad \psi' = -(2\psi + f)\varphi, \quad \eta' = -4\varphi\eta. \quad (5.6)$$

arises from the invariant reduction of the RSW equations (3.6) with the use of the subalgebra (Y_5, Y_6) . Indeed, in polar coordinates (3.2) generator Y_6 has the following form: $\hat{Y}_6 = r\partial_r + U\partial_U + V\partial_V + 2h\partial_h$. The set of basic invariants of the generators \hat{Y}_5 and \hat{Y}_6 is

$$J = \{t, r^{-1}U, r^{-1}V, r^{-2}h\}.$$

Thus, we get the following representation of the invariant solution

$$U = r\varphi(t), \quad V = r\psi(t), \quad h = r^2\eta(t). \quad (5.7)$$

Substitution of representation (5.7) into the RSW model (3.6) produces ODEs (5.6). In the general case it is difficult to get a solution of the submodel (5.6) in closed form. Let us take $\psi(t) = -f/2$. Then the second equation in (5.6) is fulfilled automatically, whereas the solution to the first and the third equations is given in an implicit form

$$\varphi = \hat{\varphi}(\eta) = \pm \sqrt{2g\eta - \frac{f^2}{4} + \left(\varphi_0^2 - 2g\eta_0 + \frac{f^2}{4}\right) \sqrt{\frac{\eta}{\eta_0}}}, \quad t(\eta) = -\frac{1}{4} \int_{\eta_0}^{\eta} \frac{d\nu}{\nu \hat{\varphi}(\nu)}. \quad (5.8)$$

Here arbitrary constants φ_0 and $\eta_0 > 0$ are values of functions $\varphi(t)$ and $\eta(t)$ at $t = 0$ (initial data for ODEs (5.6)). Function $\varphi(t)$ is strictly decreasing by virtue of the first equation (5.6) (due to $\psi = -f/2$). It follows from the last equation (5.6) that function $\eta(t)$ increases as time grows if $\varphi(t) < 0$, otherwise $\eta(t)$ decreases.

If constant $\varphi_0 \leq 0$, the minus sign precedes the root in the first formula (5.8). In fact, $\varphi(0) = \hat{\varphi}(\eta_0) \leq 0$ and $\varphi(t)$ is strictly decreasing function. It is evident from the second formula (5.8) that $t \rightarrow T_* > 0$ as $\eta \rightarrow \infty$ (integral in (5.8) converges). The solution (5.7), (5.8) can be interpreted as collapse of a liquid ring during the finite time interval $[0, T_*]$. At $t = 0$ liquid is located in the ring (or in the cylinder) $0 \leq R_{10} \leq r \leq R_{20}$. As time grows, impermeable boundaries of the ring move in accordance with the law

$$r = R_i(t) = \hat{R}_i(\eta) = R_{i0}(\eta_0/\eta)^{1/4},$$

providing the fulfillment of boundary condition $R'_i(t) = U(t, R_i(t))$. In contrast to the previous solution (5.3), (5.5) at $t \rightarrow T_*$ the circular velocity V vanishes, whereas the radial velocity U and the depth h infinitely increase. At that, the moving pistons collapse into the origin $r = 0$.

Let us take $\varphi_0 > 0$. In view of $\varphi_0 = \varphi(0) = \hat{\varphi}(\eta_0) > 0$, the plus sign precedes the root in first formula (5.8). As mentioned above, function η decreases while $\varphi > 0$. It is obvious from (5.8) that radicand vanishes at some value $\eta = \eta_1 < \eta_0$. At time $t_1 = t(\eta_1) < \infty$ function $\varphi(t)$ changes sign from positive to negative and function $\eta(t)$ grows for $t > t_1$. The behaviour of solution for $t > t_1$ is similar to the previous case $\varphi_0 \leq 0$. Thus, the solution can be interpreted as a partial spreading ($0 \leq t \leq t_1$) and collapse ($t_1 < t < T_*$) of a liquid ring.

6. Conclusions

In this paper we have studied symmetry properties and some classes of exact solutions of the RSW model using group analysis. We have determined the 9-dimensional Lie algebra of admissible infinitesimal generators. We have shown that the derived Lie algebra of symmetries is isomorphic to the Lie algebra of infinitesimal transformations admitted by (2+1)-dimensional SW equations. This allows one to employ its known optimal system of subalgebras for construction of essentially different invariant solutions of the RSW equations. Moreover, we have found change of variables which reduce the RSW equations to the SW model. This transformation allow one to construct and study solutions of the RSW equations using solutions of the SW model and vice versa. We have also derived and analysed finite transformations corresponding to the nontrivial symmetries of the RSW model. Using these transformations we have generated new time-periodic exact solutions of the RSW equations. These solutions describe fluid flow with quasi-closed particle trajectories and may be interpreted as pulsation of liquid volume under the influence of gravity and Coriolis forces. Based on some two-dimensional parameterized classes of the optimal system of subalgebras we have reduced the RSW model to the ODEs and integrate them. In particular, we have constructed and studied exact solutions of the RSW model describing rotational-symmetric stationary flows in a ring bounded by characteristics as well as time dependent flows which describe spreading and collapse of a liquid ring. We have also pointed out exact solutions of the SW equations corresponding to the obtained solutions of the RSW equations.

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